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Fermion stochastic calculus in Dirac–Fock space

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Abstract. A quantum stochastic calculus for fermions is developed where the basic integrators are based on Dirac fields and the charge operator. The associated Itô formula has seven non-trivial correction terms. Conditions are found for the solutions of stochastic differential equations to be unitary and it is shown that the corresponding quantum stochastic flow manifests a broken symmetry whereby the particle and antiparticle noises no longer balance each other. An abstract theory of such flows is then developed. By employing the unification between boson and fermion stochastic calculi, we are able to develop the entire theory using boson Fock spaces.

1. Introduction

The dissipative behaviour of a quantum system (S) is described by considering its interaction with another quantum system (R)—the 'reservoir' or 'heat bath'. We will in this paper be interested in the case where R is of a fermionic nature.

Historically, a number of models have been considered where the interaction was taken to be a 'quantum noise' (see [Acc] for a nice account of this). A precise mathematical theory of quantum noise has been developed by Hudson and Parthasarathy ([HuPa1], [Par], [Mey]) in which the behaviour of the noise is described by a stochastic calculus based on three fundamental operator-valued stochastic processes constructed from suitable annihilation, creation and number conservation operators acting in a Fock space. Specifically fermionic theories of this type were developed in [BSW], [ApHu], [App1] and [HuPa2].

Quantum stochastic calculi give rise to the following model of dissipative behaviour. Observables associated with S are modelled by (the self-adjoint) elements of some unital *-algebra $\mathfrak{A}$ acting in an 'initial' Hilbert space $\mathfrak{H}_0$ and observables associated with R act on the Hilbert space Γ (usually a Fock space). The combined state space for $(S+R)$ is $\mathfrak{H}=\mathfrak{H}_0\otimes\Gamma$. The evolution of state vectors in $\mathfrak{H}$ is described by a unitary operator valued process $U=(U(t), t\in\mathbb{R}^+)$ which satisfies an appropriate stochastic differential equation and the time evolution of an operator $x\in\mathfrak{A}$ is described by the quantum stochastic flow $J=(j_t, t\in\mathbb{R}^+)$ where $j_t(x)=U(t)xU(t)^\star$. We note that U and J do not satisfy a group law in general but are 'Markovian cocycles' in the sense of [Acc]. Finally the reduced dynamics in S is a quantum dynamical semigroup $(T_t, t\in\mathbb{R}^+)$ on $\mathfrak{A}$ [Lin] obtained through the prescription $T_t(x)=\langle\psi_0, j_t(x)\psi_0\rangle$ where ψ_0 is the vacuum vector in Γ.

A long sequence of papers by Accardi *et al* have justified the validity of this model for a number of concrete physical systems where the weak coupling or low density limit is taken—[AFL] and [AcLu] are particularly relevant to the context considered below (see also [AAFL] for an exposition of the main ideas).

So far quantum stochastic calculi have generally been based on non-relativistic free fields acting in Fock space. An exception to this rule is [FrRu] where a specific relativistic model was considered using boson noise. The aim of the present paper is to begin the study of a quantum stochastic calculus based on relativistic fermion fields where the quantum noise has both 'particle' and 'antiparticle' components (e.g. an electron-positron field).

Instead of the usual three quantum noises described above, we employ two mutually adjoint Dirac fields (sums of particle creation (annihilation) and antiparticle annihilation (creation) operators, respectively) and the charge conservation operator. We thus obtain an irreducible representation of the CARs (canonical anticommutation relations) on Dirac–Fock space which is the tensor product of two fermion Fock spaces [Tha]. Using the unification procedure of [HuPa2] and [PaSi], we are able to realise these operators in boson Fock space which leads to considerable mathematical simplification (compare [HuPa2] with [App1]). By $\mathbb{Z}_2$-grading boson Fock space in a natural way, we are also able to incorporate a $\mathbb{Z}_2$-graded initial space into this scheme so that S can be either bosonic or fermionic.

The scheme of this paper is as follows. In sections 2 and 3 below we give an account of the ideas discussed in the previous paragraph. The construction of a stochastic calculus and the vitally important Itô formula for the product of two stochastic integrals are described in section 4. In section 5 we construct the unitary process U and the associated flow J. Here we encounter an interesting phenomenon whereby the Dirac fields which drive U become decoupled in the equation for J so that the particle and antiparticle noises interact with the system observables in different ways. Finally we consider abstract 'Dirac flows' J in section 6, these being of general interest from the point of view of supersymmetric quantum theory and noncommutative differential geometry (see [Hud2], [App2]).

In order for this paper to be accessible to a wide range of potential readers, we have omitted complete proofs for some of our results where these are of a highly technical nature. Full details will appear in future publications.

Notation. Let V_1 and V_2 be complex vector spaces. $\mathfrak{L}(V_1, V_2)$ will denote the space of all linear maps from V_1 into V_2. We write this space as $\mathfrak{L}(V)$ when $V_1 = V_2 = V$. $V_1 \underline{\otimes} V_2$ will denote the algebraic tensor product of V_1 and V_2. The identity operator on V will always be denoted as I. If T is an operator in a Hilbert space any statement involving $T^{\#}$ should be read twice, once where $T^{\#}$ is read as T and once where it is read as $T^{\star}$. If $S, T \in \mathfrak{A}$ where $\mathfrak{A}$ is an algebra, $[S, T] = ST - TS$ is the commutator and $\{S, T\} = ST + TS$, the anticommutator.

2. Quantum stochastic calculus

Let K be a complex, separable, infinite-dimensional Hilbert space. We denote by $\Gamma_b(K)$ and $\Gamma_f(K)$, respectively, the boson and fermion Fock spaces over K.

Thus

$$\Gamma_b(K)=\bigoplus_{n=0}^{\infty} K^{(\otimes_s)^n} \qquad \text{and} \qquad \Gamma_f(K)=\bigoplus_{n=0}^{\infty} K^{(\otimes_a)^n}$$

where $\otimes_s$ denotes the symmetric and $\otimes_a$ the antisymmetric tensor product. For each $f\in K$, let $e(f)\in\Gamma_b(K)$ denote the exponential vector

$$e(f)=\left(1, f, \frac{f\otimes f}{\sqrt{2!}}, \ldots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots\right)$$

then $\mathscr{E}$ is dense in $\Gamma_b(K)$ where $\mathscr{E}$ is the linear span of $\{e(f), f\in K\}$. For each $f\in K$, we denote by $a(f)$ the boson annihilation operator in $\Gamma_b(K)$ and by $a^\dagger(f)$, the boson creation operator. We also introduce the conservation operator $\mathrm{d}\Gamma(X)$ for $X\in B(K)$. Precise definitions can be found in [Par]. We take $\mathscr{E}$ as a common domain for all three classes of operator. Note that on $\mathscr{E}$ we have

$$a(f)^\star=a^\dagger(f) \qquad \text{and} \qquad \mathrm{d}\Gamma(X)^\star=\mathrm{d}\Gamma(X^\star)$$

for each $f\in K, X\in B(K)$. We further have that the extended CCRs hold on $\mathscr{E}$ i.e. for all $f, g\in K$ and X, $Y\in B(K)$

$$[a(f), a(g)]=[a^\dagger(f), a^\dagger(g)]=0$$

$$[a(f), a^\dagger(g)]=\langle f, g\rangle I$$

$$[\mathrm{d}\Gamma(X), \mathrm{d}\Gamma(Y)]=\mathrm{d}\Gamma([X, Y])$$

$$[a(f), \mathrm{d}\Gamma(X)]=a(X^\star f)$$

$$[\mathrm{d}\Gamma(X), a^\dagger(f)]=-a^\dagger(Xf).$$

Now let P be a continuous projection valued measure on $\mathbb{R}$ taking values in $B(K)$. We write $P_t=P(-\infty, t)$ for each $t\in\mathbb{R}$. We fix u, $v\in K$ and $X\in B(K)$ such that $[X, P_t]=0$ for all $t\in\mathbb{R}$. We define the boson annihilation process $A_u=\{A_u(t), t\in\mathbb{R}\}$ by

$$A_u(t)=a(P_t u).$$

The boson creation process $A_v^\dagger=\{A_v^\dagger(t), t\in\mathbb{R}\}$ is given by

$$A_v^\dagger(t)=a^\dagger(P_t v)$$

and the conservation process $\Lambda_X=\{\Lambda_X(t), t\in\mathbb{R}\}$ is

$$\Lambda_X(t)=\mathrm{d}\Gamma(P_t X).$$

In [Par] quantum stochastic integrals $M=\{M(t), t\in\mathbb{R}\}$ of the form

$$M(t)=\int_{-\infty}^{t} (H_1(s)\,\mathrm{d}A_v^\dagger(s)+H_2(s)\,\mathrm{d}\Lambda_X(s)+H_3(s)\,\mathrm{d}A_u(s)+H_4(s)\,\mathrm{d}s) \tag{2.1}$$

are defined as families of linear operators with domain $\mathscr{E}$ where $H_j=\{H_j(t), t\in\mathbb{R}\}$ for $j=1, 2, 3, 4$ are suitable operator-valued processes.

A vital role in this theory is played by the quantum Itô formula which states that if M_1 and M_2 are two stochastic integrals of the form (2.1) then so is their product

$M_1M_2=\{M_1(t)M_2(t), t\in\mathbb{R}\}$. Moreover the form of the product is determined by

$$d(M_1M_2)=dM_1\cdot M_2+M_1\cdot dM_2+dM_1\cdot dM_2. \tag{2.2}$$

The Itô correction term $dM_1\cdot dM_2$ is computed by bilinear extension of the rule that all products of differentials vanish with the exception of

$$dA_u(t)\cdot dA_v^\dagger(t)=d\langle P_t u, v\rangle$$

$$d\Lambda_X(t)\cdot dA_v^\dagger(t)=dA_{Xv}^\dagger(t)$$

$$dA_u(t)\cdot d\Lambda_X(t)=dA_{X^*u}(t)$$

$$d\Lambda_X(t)\cdot d\Lambda_X(t)=d\Lambda_{X^2}(t).$$

Now let $J=\{J(t), t\in\mathbb{R}\}$ be defined on $\mathscr{E}$ by

$$J(t)e(f)=e((I-2P_t)f) \qquad \text{for each } f\in K$$

so that each $J(t)$ extends to a self-adjoint, unitary operator on $\Gamma_b(K)$ which thus satisfies $J(t)^2=I$ for each $t\in\mathbb{R}$. We further define for each $u, v\in K$,

$$b(u)=\int_{-\infty}^{\infty} J(t)\, dA_u(t) \qquad b^\dagger(v)=\int_{-\infty}^{\infty} J(t)\, dA_v^\dagger(t).$$

It is shown in [PaSi] and [HuPa2] that we thus obtain an irreducible representation of the CARs in $\Gamma_b(K)$. In fact the extended CARs hold with the same conservation operators as for the boson case, i.e.

$$\{b(u), b(v)\}=\{b^\dagger(u), b^\dagger(v)\}=0$$

$$\{b(u), b^\dagger(v)\}=\langle u, v\rangle I$$

$$[b(u), d\Gamma(X)]=b(X^* u)$$

$$[d\Gamma(X), b^\dagger(u)]=-b^\dagger(Xu).$$

This representation can be used to construct a canonical isomorphism ι_K between $\Gamma_b(K)$ and $\Gamma_f(K)$, the details of which can be found in [PaSi].

3. Dirac–Fock space

Let $\mathfrak{H}$ be a complex, separable Hilbert space. It is said to be $\mathbb{Z}_2$-*graded* if it has a decomposition

$$\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-.$$

$\mathfrak{H}_+$ is called the odd sector and $\mathfrak{H}_-$, the even sector of $\mathfrak{H}$. A dense linear manifold $\mathscr{D}$ is called a $\mathbb{Z}_2$-*graded domain* if it admits the vector space decomposition $\mathscr{D}=\mathscr{D}_+\oplus\mathscr{D}_-$ wherein $\mathscr{D}_+\subseteq\mathfrak{H}_+$ and $\mathscr{D}_-\subseteq\mathfrak{H}_-$. We denote by θ the parity operator in $\mathfrak{H}$ which acts as I on $\mathfrak{H}_+$ and $-I$ on $\mathfrak{H}_-$. θ is self-adjoint and unitary.

A linear operator T in $\mathfrak{H}$ with domain $\mathscr{D}$ is said to be *even* if $T\mathscr{D}_\pm\subseteq\mathfrak{H}_\pm$ and *odd* if $T\mathscr{D}_\pm\subseteq\mathfrak{H}_\mp$. The parity *-automorphism ρ of $B(\mathfrak{H})$ is defined by $\rho(T)=\theta T\theta$.

If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are $\mathbb{Z}_2$-graded Hilbert spaces, their tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is also $\mathbb{Z}_2$-graded by the prescription

$$(\mathfrak{H}_1 \otimes \mathfrak{H}_2)_+ = (\mathfrak{H}_{1+} \otimes \mathfrak{H}_{2+}) \oplus (\mathfrak{H}_{1-} \otimes \mathfrak{H}_{2-})$$

$$(\mathfrak{H}_1 \otimes \mathfrak{H}_2)_- = (\mathfrak{H}_{1+} \otimes \mathfrak{H}_{2-}) \oplus (\mathfrak{H}_{1+} \otimes \mathfrak{H}_{2-}).$$

Similarly if $\mathscr{D}_j$ are $\mathbb{Z}_2$-graded domains in $\mathfrak{H}_j$ $(j=1,2)$, then $\mathscr{D}_1 \underline{\otimes} \mathscr{D}_2$ is a $\mathbb{Z}_2$-graded domain in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. If T_j are linear operators in $\mathfrak{H}_j$ with domain $\mathscr{D}_j$ $(j=1,2)$ and T_2 is of definite parity, the *Chevalley tensor product* $T_1 \hat{\otimes} T_2$ is defined by linear extension of the formula

$$(T_1 \hat{\otimes} T_2)(u_1 \otimes u_2) = (-1)^{\delta(T_2)\delta(u_1)}(T_1 u_1 \otimes T_2 u_2)$$

where $u_j \in \mathscr{D}_j$ $(j=1,2)$ with u_1 of definite parity and

$$\delta(T_2) = \begin{cases} 1 & \text{if } T_2 \text{ is odd} \\ 0 & \text{if } T_2 \text{ is even} \end{cases}$$

with $\delta(u_1)$ defined similarly (see [Che], [ApHu]). The definition extends by linearity to the case where T_2 is not of definite parity. We note the following easily verified properties of the Chevalley tensor product:

(i) $(S_1 \hat{\otimes} S_2)(T_1 \hat{\otimes} T_2) = (-1)^{\delta(S_2)\delta(T_1)}(S_1 S_2 \hat{\otimes} T_1 T_2)$

for S_2, T_1 of definite parity, the operators being such that the right hand side is well-defined.

(ii) $\{S \hat{\otimes} I, I \hat{\otimes} T\} = 0$ if S and T are both odd.
$[S \hat{\otimes} I, I \hat{\otimes} T] = 0$ if S and T are of definite parity but not both odd.

(iii) $(S \hat{\otimes} T)^\star = -S^\star \hat{\otimes} T^\star$ if S and T are both odd.

Now consider boson Fock space $\Gamma_b(K)$—this is $\mathbb{Z}_2$-graded by the prescription

$$\Gamma_b(K)_+ = \bigoplus_{n=0}^{\infty} K^{(\otimes_s)^{2n}} \qquad \text{and} \qquad \Gamma_b(K)_- = \bigoplus_{n=0}^{\infty} K^{(\otimes_s)^{2n+1}}.$$

Define for each $f \in K$,

$$\cosh(f) = \left(1, \frac{f^{\otimes^2}}{\sqrt{2!}}, \ldots, \frac{f^{\otimes^{2n}}}{\sqrt{2n!}}, \ldots\right) \in \Gamma_b(K)_+$$

$$\sinh(f) = \left(f, \frac{f^{\otimes^3}}{\sqrt{3!}}, \ldots, \frac{f^{\otimes^{2n+1}}}{\sqrt{(2n+1)!}}, \ldots\right) \in \Gamma_b(K)_-$$

then $\mathscr{E}$ is a $\mathbb{Z}_2$-graded domain in $\Gamma_b(K)$ with $\mathscr{E}_+$ being the linear span of $\{\cosh(f), f \in \mathfrak{H}\}$ and $\mathscr{E}_-$, the linear span of $\{\sinh(f), f \in \mathfrak{H}\}$. We note that boson annihilation and creation operators are odd and conservation operators are even.

Fermion Fock space $\Gamma_f(K)$ is also $\mathbb{Z}_2$-graded by

$$\Gamma_f(K)_+ = \bigoplus_{n=0}^{\infty} K^{(\otimes_a)^{2n}} \qquad \text{and} \qquad \Gamma_f(K)_- = \bigoplus_{n=0}^{\infty} K^{(\otimes_a)^{2n+1}}.$$

We note that the isomorphism ι_K preserves the grading i.e. $\iota_K(\Gamma_b(K)_\pm) = \Gamma_f(K)_\pm$.

Now suppose that K is itself $\mathbb{Z}_2$-graded, $K=K_+\oplus K_-$ and denote by $\pi_\pm$ the orthogonal projections from K onto $K_\pm$, then we have the canonical isomorphism

$$\Gamma_b(K)\simeq\Gamma_b(K_+)\otimes\Gamma_b(K_-)$$

wherein each $e(f)$ is mapped to $e(\pi_+f)\otimes e(\pi_-f)$. We use this isomorphism to identify the two spaces. Let $\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-$ be another $\mathbb{Z}_2$-graded Hilbert space and define $K_+=\mathfrak{H}_+$ and $K_-=C\mathfrak{H}_-$ where C is complex conjugation. We define *Dirac-Fock space* $\mathfrak{F}(\mathfrak{H})$ to be the $\mathbb{Z}_2$-graded Hilbert space $\Gamma_b(K)$ where K is as above. Using the isomorphism $\iota_{K_+}\otimes\iota_{K_-}$, we identify $\mathfrak{F}(\mathfrak{H})$ with $\Gamma_f(K_+)\otimes\Gamma_f(K_-)$ (this latter form may be more familiar to some readers—see e.g. [Tha]).

Now for each $u\in\mathfrak{H}$, define the *Dirac field operators* by

$$\Psi(u)=b(\pi_+u)\,\hat{\otimes}\,I+I\,\hat{\otimes}\,b^\dagger(\pi_-\bar{u})$$

$$\Psi^\dagger(u)=b^\dagger(\pi_+u)\,\hat{\otimes}\,I+I\,\hat{\otimes}\,b(\pi_-\bar{u})$$

and for $X\in B(\mathfrak{H})$,

$$X=\begin{pmatrix}X_+ & 0\\ 0 & X-\end{pmatrix}$$

define the *charge operator*

$$\Xi(X)=\mathrm{d}\Gamma(X_+)\,\hat{\otimes}\,I-I\,\hat{\otimes}\,\mathrm{d}\Gamma(X_-^\star).$$

It is shown in [Tha] that $\{\Psi(u),\Psi^\dagger(v);u,v\in\mathfrak{H}\}$ yield an irreducible representation of the CARs in $\mathfrak{F}(\mathfrak{H})$. Moreover, we note that the extended CARs hold i.e.

$$\{\Psi(u),\Psi(v)\}=\{\Psi^\dagger(u),\Psi^\dagger(v)\}=0$$

$$\{\Psi(u),\Psi^\dagger(v)\}=\langle u,v\rangle I$$

$$[\Xi(X),\Xi(Y)]=\Xi([X,Y])$$

$$[\Psi(u),\Xi(X)]=\Psi(X^\star u)$$

$$[\Xi(X),\Psi^\dagger(u)]=-\Psi^\dagger(Xu)$$

for all $u,v\in\mathfrak{H}$, $X,Y\in B(\mathfrak{H}_+)\oplus B(\mathfrak{H}_-)$.

In the following we will occasionally use the notation $p^\#(f)=b^\#(\pi_+f)\,\hat{\otimes}\,I$ to denote fermion *particle* creation and annihilation operators and $a^\#(f)=I\,\hat{\otimes}\,b^\#(\pi_-\bar{f})$ to denote *antiparticle* creation and annihilation operators.

We remark that we have the following stochastic integral representations for Dirac fields in terms of boson processes

$$\Psi(u)=\int_{-\infty}^{\infty}J(t)\,\mathrm{d}A_{\pi_+u}(t)\hat{\otimes}I+I\hat{\otimes}\int_{-\infty}^{\infty}J(t)\,\mathrm{d}A^\dagger_{\pi_-\bar{u}}(t)\tag{3.1}$$

$$\Psi^\dagger(u)=\int_{-\infty}^{\infty}J(t)\,\mathrm{d}A^\dagger_{\pi_+u}(t)\hat{\otimes}I+I\hat{\otimes}\int_{-\infty}^{\infty}J(t)\,\mathrm{d}A_{\pi_-\bar{u}}(t)\tag{3.2}$$

$$\Xi(X)=\int_{-\infty}^{\infty}\mathrm{d}\Lambda_{X_+}(t)\hat{\otimes}I-I\hat{\otimes}\int_{-\infty}^{\infty}\mathrm{d}\Lambda_{X_-^\star}(t).\tag{3.3}$$

Here J is defined with respect to a $B(\mathfrak{H})$-valued projection valued measure P which satisfies $[P(t),\pi_\pm]=0$ for all $t\in\mathbb{R}$.

4. Stochastic integration

In the remainder of this paper, in order to develop a stochastic theory, our projection valued measure P will be defined only on $\mathbb{R}^+$.

We define the *Dirac processes* $\Psi_u^\# = \{\Psi_u^\#(t), t \in \mathbb{R}^+\}$ by

$$\Psi_u^\# = \Psi^\#(P(t)u) \qquad \text{for} \qquad t \in \mathbb{R}^+, u \in \mathfrak{H}.$$

So that each

$$\Psi_u(t) = \mathscr{P}_u(t) + \mathscr{A}_u^\dagger(t)$$

and

$$\Psi_u^\dagger(t) = \mathscr{P}_u^\dagger(t) + \mathscr{A}_u(t)$$

where $\mathscr{P}_u^\#(t) = p^\#(P(t)u)$ and $\mathscr{A}_u^\#(t) = a^\#(P(t)u)$. We also define the *charge process* $\Xi_X = \{\Xi_X(t), t \in \mathbb{R}^+\}$ by

$$\Xi_X(t) = \Xi(P(t)X)$$

for $X \in B(\mathfrak{H}_+) \oplus B(\mathfrak{H}_-)$. By (3.1)–(3.3), we have that

$$\mathrm{d}\Psi_u(t) = J(t)\, \mathrm{d}A_{\pi_+ u}(t) \hat{\otimes} I + I \hat{\otimes} J(t)\, \mathrm{d}A_{\pi_- \bar{u}}^\dagger(t) \tag{4.1}$$

$$\mathrm{d}\Psi_u^\dagger(t) = J(t)\, \mathrm{d}A_{\pi_+ u}^\dagger(t) \hat{\otimes} I + I \hat{\otimes} J(t)\, \mathrm{d}A_{\pi_- \bar{u}}(t) \tag{4.2}$$

$$\mathrm{d}\Xi_X(t) = \mathrm{d}\Lambda_{X_+}(t) \hat{\otimes} I - I \hat{\otimes} \mathrm{d}\Lambda_{X_-^*}(t). \tag{4.3}$$

We may now consider stochastic integrals of the form $M = \{M(t), t \in \mathbb{R}^+\}$ where

$$M(t) = \int_{-\infty}^{t} (\mathrm{d}\Psi_u^\dagger(s) H_1(s) + H_2(s)\, \mathrm{d}\Xi_X(s) + H_3(s)\, \mathrm{d}\Psi_v(s) + H_4(s)\, \mathrm{d}s)$$

where $H_j = \{H_j(t), t \in \mathbb{R}^+\}$, $(j = 1, 2, 3, 4)$ are suitable operator-valued processes in $\mathfrak{F}(\mathfrak{H})$. Now let $M_j = \{M_j(t), t \in \mathbb{R}^+\}, j = 1, 2$ be two stochastic integrals of the same type. In order to get a workable Itô table we make the following assumption

$$\begin{aligned} &X = X^\star \\ &X_+^2 = X_+ \qquad X_-^2 = -X_- \\ &X_+\pi_+ = \pi_+ \qquad X_-\pi_- = -\pi_-. \end{aligned} \tag{4.4}$$

We will see below that these are quite natural conditions. We then obtain the following Itô formula

$$\mathrm{d}(M_1 M_2) = \mathrm{d}M_1\, M_2 + M_1\, \mathrm{d}M_2 + \mathrm{d}M_1\, \mathrm{d}M_2 \tag{4.5}$$

where the Itô correction term is calculated (subject to parity considerations, see e.g. [App1]) by bilinear extension of the rules

$$\mathrm{d}\Psi_u(t)\, \mathrm{d}\Psi_v^\dagger(t) = \mathrm{d}\langle P(t)\pi_+ u, \pi_+ v\rangle$$

$$\mathrm{d}\Psi_u^\dagger(t)\, \mathrm{d}\Psi_v(t) = \mathrm{d}\langle P(t)\pi_- u, \pi_- v\rangle$$

$$\mathrm{d}\Xi_X(t)\, \mathrm{d}\Xi_X(t) = \mathrm{d}\Xi_X(t)$$

$$\mathrm{d}\Xi_X(t)\, \mathrm{d}\Psi_u(t) = \mathrm{d}\mathscr{A}_u^\dagger(t)$$

$$d\Psi_u(t)\, d\Xi_X(t) = d\mathscr{P}_u(t)$$

$$d\Xi_X(t)\, d\Psi_u^\dagger(t) = d\mathscr{P}_u^\dagger(t)$$

$$d\Psi_u^\dagger(t)\, d\Xi_X(t) = d\mathscr{A}_u(t).$$

These can all be calculated from (2.1) using (4.1)-(4.3) and assumption (4.4).

Example. Let $\mathfrak{H} = L^2(\mathbb{R}^+, V) \simeq L^2(\mathbb{R}^+) \otimes V$ where V is the $\mathbb{Z}_2$-graded Hilbert space $V = V_+ \oplus V_-$. In this case we take

$$K_+ = L^2(\mathbb{R}^+, V_+) \qquad \text{and} \qquad K_- = L^2(\mathbb{R}^+, \bar{V}_-).$$

We put

$$P_t(f \otimes u) = \chi_{[0,t)} f \otimes u$$

where if A is a measurable set in $\mathbb{R}^+$, χ_A is the indicator function

$$\chi_A(p) = 1 \quad \text{if} \quad p \in A \qquad \chi_A(p) = 0 \quad \text{if} \quad p \notin A.$$

We take X to be the parity operator in $\mathfrak{H}$ i.e.

$$X = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{so that} \qquad \pi_\pm = \tfrac{1}{2}(I \pm X)$$

then it is easy to see that (4.4) is satisfied.

In the following, we will always work in this context. (To make direct contact with relativistic quantum field theory, we might take $V = L^2(\mathbb{R}^3, \mathbb{C}^4) \simeq L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.) To simplify the Itô formula (4.5), we will from now on take $v = u$ with $\|u\| = 1$. We then find that we have

$$d\Psi_u(t)\, d\Psi_v^\dagger(t) = \lambda^2\, dt$$

$$d\Psi_u^\dagger(t)\, d\Psi_v(t) = \mu^2\, dt$$

where $\lambda^2 = \|\pi_+ u\|^2$ and $\mu^2 = \|\pi_- u\|^2$, so that

$$\lambda^2 + \mu^2 = 1.$$

These are reminiscent of the Itô correction terms arising from stochastic calculi based on quasi-free states of the CARs (see e.g. [ApFr]).

Note. For previously studied examples of quantum stochastic calculi, the simplest case has always been obtained by taking $V = \mathbb{C}$. Observe that if we make such a choice in this case we return to the usual fermion stochastic calculus in the context of [HuPa2].

5. Unitary evolutions

In this section we introduce another $\mathbb{Z}_2$-graded Hilbert space $\mathfrak{H}_0$ and work in the $\mathbb{Z}_2$-graded tensor product

$$\mathscr{H} = \mathfrak{H}_0 \otimes \mathfrak{F}(L^2(\mathbb{R}^+, V)).$$

We identify all linear operators L in $\mathfrak{H}_0$ with their applications $L\hat{\otimes} I$ to the whole of $\mathscr{H}$ and similarly identify linear operators M in $\mathfrak{F}(L^2(\mathbb{R}^+, \mathfrak{H}))$ with $I\hat{\otimes} M$ on $\mathscr{H}$. In the following, ρ will always denote the parity *-automorphism in $B(\mathfrak{H}_0)$.

All results about stochastic integrals discussed in the previous section extend to this context, with obvious modifications. Now let $L_j, j=1, 2, 3, 4$ be densely defined linear operators in $\mathfrak{H}_0$ with common invariant $\mathbb{Z}_2$-graded domain $\mathfrak{D}_0$. We consider the quantum stochastic differential equation

$$\mathrm{d}U = U(\mathrm{d}\Psi_u^\dagger L_1 + L_2\,\mathrm{d}\Xi_X + L_3\,\mathrm{d}\Psi_u + L_4\,\mathrm{d}t) \tag{5.1}$$

with initial condition $U(0)=I$.

We will assume that (5.1) has a unique solution—the details of the proof will be given elsewhere. We remark that the case where the L_j's are bounded follows by a similar argument to that of theorem 5.1 of [HuPa2]. The unbounded case, subject to certain analytical constraints on the L_j's can be solved using various techniques (see e.g. [Fag] and [Moh] for precise details or chapter 6 of [Mey] for a nice introductory account, all with respect to the boson case).

We are interested in the case where the solution $U=(U(t), t\in\mathbb{R}^+)$ is such that each $U(t)$ is a unitary operator. We then say that U is a *unitary process*. Following [Hud2], we impose the requirement that each $U(t)$ is even. This has the consequence that L_1 and L_3 are odd with L_2 and L_4 being even. We then obtain the following.

Theorem 5.1. A necessary and sufficient condition for U to be a unitary process is that there exists an even unitary operator W in $\mathfrak{H}_0$, an even self-adjoint operator H in $\mathfrak{H}_0$ and an odd operator L in $\mathfrak{H}_0$ satisfying

$$[L^\star, W]=0 \tag{5.2}$$

with

$$L_1 = L$$

$$L_2 = W - I$$

$$L_3 = -L^\star W$$

$$L_4 = \mathrm{i}H - \tfrac{1}{2}\lambda^2 L^\star L - \tfrac{1}{2}\mu^2 LL^\star.$$

Proof. The argument is standard (see e.g. [Par]) and for simplicity we will prove only the necessity part here. First suppose that each $U(t)$ is isometric so that $U(t)^\star U(t)=I$. By (4.5), we obtain

$$\mathrm{d}U^\star\, U + U^\star \mathrm{d}U + \mathrm{d}U^\star\,\mathrm{d}U = 0 \tag{$*$}$$

where we note by (5.1), we have

$$\mathrm{d}U^\star = (\mathrm{d}\Psi_u^\dagger L_3^\star + L_2^\star\,\mathrm{d}\Xi_X + L_1^\star\,\mathrm{d}\Psi_u + L_4^\star\,\mathrm{d}t)U^\star.$$

Substituting into $(*)$ yields

$$(\mathrm{d}\Psi_u^\dagger L_3^\star + L_2^\star\,\mathrm{d}\Xi_X + L_1^\star\,\mathrm{d}\Psi_u + L_4^\star\,\mathrm{d}t) + (\mathrm{d}\Psi_u^\dagger L_1 + L_2\,\mathrm{d}\Xi_X + L_3\,\mathrm{d}\Psi_u + L_4\,\mathrm{d}t)$$
$$+ (L_2^\star L_2\,\mathrm{d}\Xi_X + \lambda^2 L_1^\star L_1\,\mathrm{d}t + \mu^2 L_3^\star L_3\,\mathrm{d}t + L_2^\star L_3\,\mathrm{d}\mathscr{A}_u^\dagger$$
$$+ L_1^\star L_2\,\mathrm{d}\mathscr{P}_u + \mathrm{d}\mathscr{P}_u^\star L_2^\star L_1 + \mathrm{d}\mathscr{A}_u L_3^\star L_2) = 0.$$

Equating coefficients yields

$$\mathrm{d}\Xi_X : L_2 + L_2^\star + L_2^\star L_2 = 0 \tag{i}$$

$$\mathrm{d}\mathscr{P}_u : L_1^\star + L_3 + L_1^\star L_2 = 0 \tag{ii}$$

$$\mathrm{d}\mathscr{A}_u^\dagger : L_1^\star + L_3 + L_2^\star L_3 = 0 \tag{iii}$$

$$\mathrm{d}t : L_4 + L_4^\star + \lambda^2 L_1^\star L_1 + \mu^2 L_3^\star L_3 = 0. \tag{iv}$$

(We have omitted the coefficients of $\mathrm{d}\mathscr{P}_u^\dagger$ and $\mathrm{d}\mathscr{A}_u$ as these are just the adjoints of (iii) and (ii) respectively.)

From (i), we have that $L_2 = W - I$, where W is an isometry. Putting $L_1 = L$, we find that

$$\text{(ii)} \Rightarrow L_3 = -L^\star W \qquad \text{and} \qquad \text{(iii)} \Rightarrow L_3 = -WL^\star$$

(5.2) ensures that these are equal. Finally (iv) yields the required form for L_4.

Stochastically differentiating the condition $U(t)U(t)^\star = I$ yields the additional condition that W is co-isometric. □

Given such a unitary process U, we define an *even quantum stochastic flow* $J = (j_t, t \in \mathbb{R}^+)$ on the $\mathbb{Z}_2$-graded *-algebra $B(\mathfrak{H}_0)$ by

$$j_t(x) = U(t) x U(t)^\star \tag{5.3}$$

where $x \in B(\mathfrak{H}_0)$, $t \in \mathbb{R}^+$.

A standard exercise in the use of (4.5) yields the following differential form of (5.3)

$$\begin{aligned} \mathrm{d}j_t(x) = \mathrm{d}\mathscr{P}_u^\dagger j_t(\alpha(x)) + \mathrm{d}\mathscr{A}_u j_t(\beta(x) W^\star) + j_t(\lambda(x))\, \mathrm{d}\Xi_X \\ + j_t(\tilde{\alpha}(x))\, \mathrm{d}\mathscr{P}_u + j_t(W\tilde{\beta}(x))\, \mathrm{d}\mathscr{A}_u^\dagger + j_t(\tau(x))\, \mathrm{d}t \end{aligned} \tag{5.4}$$

where

$$\alpha(x) = Lx - W\rho(x) W^\star L \qquad \beta(x) = Lx - \rho(x) L \qquad \lambda(x) = WxW^\star - x$$

$$\tilde{\alpha}(x) = \alpha(x^\star)^\star \qquad \tilde{\beta}(x) = \beta(x^\star)^\star$$

$$\begin{aligned} \tau(x) = \mathrm{i}[H, x] - \tfrac{1}{2}\lambda^2 \{L^\star L x - 2L^\star W \rho(x) W^\star L + x L^\star L\} \\ - \tfrac{1}{2}\mu^2 \{L L^\star x - 2L\rho(x) L^\star + x L L^\star\}. \end{aligned}$$

We note that the prescription $\langle e(0), j_t(\,\cdot\,) e(0)\rangle$ yields a quantum dynamical semigroup on $B(\mathfrak{H}_0)$ with infinitesimal generator τ.

It is interesting to compare the forms of (5.1) and (5.4). Equation (5.1) (under the conditions of theorem 5.1) describes the evolution of states of quantum system (as described by $\mathfrak{H}_0$) coupled to an external fermion field (described by $\mathfrak{F}(\mathfrak{H})$). In (5.1) there is complete symmetry between the particle and antiparticle sectors of this field. Equation (5.4) however describes the corresponding evolution of observables. Here we find that the symmetry between particles and antiparticles is broken. Indeed particle creation is coupled to the system by the twisted superderivation α and antiparticle annihilation is coupled by the doubly twisted superderivation γ where $\gamma(\,\cdot\,) = \beta(\,\cdot\,) W^\star$ (see below, lemma 6.1).

It is tempting to speculate that similar processes to the above may be responsible for the excess of particles over antiparticles in the observed universe.

We note that symmetry is restored in (5.4) (i.e. $\gamma = \alpha$) if and only if $W = I$ in which case the $d\Xi$ term is absent in both (5.1) and (5.4).

6. Dirac flows on superalgebras

Let $\mathfrak{A} \subseteq B(\mathfrak{H}_0)$ be a $\mathbb{Z}_2$-graded unital *-algebra such that the grading on $\mathfrak{A}$ is compatible with that on $\mathfrak{H}_0$ (i.e. $\rho(x) = \theta x \theta$, where θ is the parity operator on $\mathfrak{H}_0$). In this section, we aim to generalize the flow of (5.3), by replacing $B(\mathfrak{H}_0)$ by $\mathfrak{A}$ and following the ideas of [Hud1, 2] and [App2]. Let $J = \{j_t, t \in \mathbb{R}^+\}$ be a family of *-homomorphisms from $\mathfrak{A}$ into $B(\mathfrak{H})$. We say that J is a *Dirac flow* on $\mathfrak{A}$ if the following conditions are satisfied for each $x \in \mathfrak{A}$

(i) $j_0(x) = x \hat{\otimes} I$

(ii) Each j_t is even i.e. $j_t(\rho(x)) = \rho'(j_t(x))$ where ρ' is the parity *-automorphism on $B(\mathfrak{H})$ for all $t \in \mathbb{R}^+$,

(iii) There exist $\lambda, \alpha, \gamma, \tilde{\alpha}, \tilde{\gamma} \in \mathfrak{L}(\mathfrak{A})$ and $\tau \in \mathfrak{L}(\mathfrak{A}, \mathfrak{L}(\mathfrak{A}))$ such that

$$dj_t(x) = d\mathscr{P}_u^\dagger j_t(\alpha(x)) + d\mathscr{A}_u j_t(\gamma(x)) + j_t(\lambda(x))\, d\Xi_X$$
$$+ j_t(\tilde{\alpha}(x))\, d\mathscr{P}_u + j_t(\tilde{\gamma}(x))\, d\mathscr{A}_u^\dagger + j_t(\tau(x))\, dt. \tag{6.1}$$

Using the facts that $j_t(I) = I, j_t(x^\star) = j_t(x)^\star$ and $j_t(xy) = j_t(x) j_t(y)$ for all $x, y \in \mathfrak{U}$, $t \in \mathbb{R}^+$ and (ii) above, we deduce the following properties of the 'structure maps':

(S1) $\lambda(I) = \alpha(I) = \tilde{\alpha}(I) = \gamma(I) = \tilde{\gamma}(I) = \tau(I) = 0$
(S2) λ and τ are even, α, $\tilde{\alpha}$, γ and $\tilde{\gamma}$ are odd,
(S3) $\lambda(x)^\star = \lambda(x^\star)$, $\tau(x)^\star = \tau(x^\star)$
$\tilde{\alpha}(x) = \alpha(x^\star)^\star$, $\tilde{\gamma}(x) = \gamma(x^\star)^\star$
(S4) $\lambda = \sigma - \mathrm{id}$ where σ is an even identity preserving *-endomorphism of $\mathfrak{A}$,
(S5) $\alpha(xy) = \alpha(x)y + \phi(x)\alpha(y)$ where $\phi = \sigma \circ \rho$.
(We say that α is a super ϕ-derivation.)
(S6) $\gamma(xy) = \gamma(x)\sigma(y) + \rho(x)\gamma(y)$.
(We say that γ is a super (σ, ρ)-derivation).
(S7) $(\Delta\tau)(x, y) = -\lambda^2 \tilde{\alpha}(x)\alpha(y) - \mu^2 \gamma(\rho(x))\tilde{\gamma}(\rho(y))$
where $(\Delta\tau)(x, y) = \tau(x)y - \tau(xy) + x\tau(y)$

i.e. Δ is the Hochshild coboundary operator for the complex of multilinear maps from $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{A})$.

Equation (5.3) gives an example of an inner Dirac flow with $\mathfrak{A} = B(\mathfrak{H}_0)$. In that case we have $\sigma(x) = WxW^\star$. The relationship between γ and β is clarified by the following.

Lemma 6.1. Let $w \in \mathfrak{A}$ be even and invertible and let β be a superderivation on $\mathfrak{A}$ i.e. for all $x, y \in \mathfrak{A}$

$$\beta(xy) = \beta(x)y + \rho(x)\beta(y).$$

Define $\gamma(x) = \beta(x)w^{-1}$, then γ is a super (σ, ρ)-derivation where $\sigma(x) = wxw^{-1}$.

Proof. $\gamma(xy) = \beta(xy)w^{-1}$

$$= (\beta(x)y + \rho(x)\beta(y))w^{-1}$$
$$= \beta(x)w^{-1}wyw^{-1} + \rho(x)\beta(y)w^{-1}$$
$$= \gamma(x)\sigma(y) + \rho(x)\gamma(y).$$ □

In general, irrespective of the analytical problems involved, there may be algebraic obstructions to the construction of Dirac flows which are not inner as in (5.3). More precisely, given σ, α and γ *there is no guarantee that* τ *exists satisfying* (S7). We close this section by indicating how to solve this problem under the assumption that σ is a *-automorphism of $\mathfrak{A}$.

We need two results from [App2].

(a) [App2—lemma 2.1]

If ε is a super σ-derivation, then $\hat{\varepsilon}$ is a super σ^{-1}-derivation where

$$\hat{\varepsilon} = \varepsilon \circ \sigma^{-1}.$$

(b) [App2—theorem 2.2].

Define $T_\varepsilon \in \mathfrak{L}(\mathfrak{A}, \mathfrak{L}(\mathfrak{A}))$ by

$$T_\varepsilon(x) = \tfrac{1}{2}(\hat{\varepsilon}\varepsilon x - 2\hat{\varepsilon}\phi(x)\varepsilon - x\hat{\varepsilon}\varepsilon)$$

where $x \in \mathfrak{A}$ and $\phi = \sigma \circ \rho$, then for all $x, y \in \mathfrak{A}$

$$(\Delta T_\varepsilon)(x, y) = -\tilde{\varepsilon}(x)\varepsilon(y).$$

Before proving our main result we need the following lemma.

Lemma 6.2. Let $\omega = \gamma \circ \rho$ then $\tilde{\omega}$ is a super ϕ-derivation on $\mathfrak{A}$

Proof. For $x, y \in \mathfrak{A}$, we have

$$\omega(ab) = \gamma(\rho(ab)) = \gamma(\rho(a)\rho(b))$$
$$= \rho^2(a)\gamma(\rho(b)) + \gamma(\rho(a))\sigma(\rho(b)) \qquad \text{by (S6)}$$
$$= a\omega(b) + \omega(a)\phi(b).$$

Hence

$$\tilde{\omega}(ab) = \omega(b^\star a^\star)^\star$$
$$= \tilde{\omega}(a)b + \phi(a)\tilde{\omega}(b) \qquad \text{as required.}$$ □

Theorem 6.2. Define $\tau \in \mathfrak{L}(\mathfrak{A}, \mathfrak{L}(\mathfrak{A}))$ by

$$\tau = \lambda^2 T_\alpha + \mu^2 T_{\tilde{\omega}} + \delta$$

where δ is a *-derivation on $\mathfrak{A}$ then (S7) is satisfied, i.e.

$$(\Delta\tau)(x, y) = -\lambda^2\tilde{\alpha}(x)\alpha(y) - \mu^2\gamma(\rho(x))\tilde{\gamma}(\rho(y))$$

for all $x, y \in \mathfrak{A}$.

Proof. By (b) above, we have

$$(\Delta T_\alpha)(x, y) = -\tilde{\alpha}(x)\alpha(y).$$

By (b) again and lemma 6.2

$$(\Delta T_{\tilde{\omega}})(x, y) = -\tilde{\tilde{\omega}}(x)\omega(y).$$

However $\tilde{\tilde{\omega}} = \omega = \gamma \circ \rho$ and the required result follows by linearity of Δ. □

In the case where $\mathfrak{A}$ is a norm-dense *-subalgebra of a $C^\star$-algebra, a scheme for constructing a large class of Dirac flows by unitary conjugation can be obtained by a slight perturbation of the procedure discussed in section 4 of [App2].

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